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CLASSES OF NONPARAMETRIC TESTS FOR TWO-SAMPLE SCALE PROBLEM

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Abstract

Two classes of nonparametric tests based on extremes of the subsamples are proposed for the two-sample scale problem and their asymptotic distributions are derived. The tests are based on U-statistics with kernels being functions of maxima or minima of the positive or negative observations of the subsamples of specified sizes. The performances of the classes of tests are discussed in terms of Pitman asymptotic relative efficiency (ARE) and their empirical power. An example is discussed for illustration of the application of the tests.

1. Introduction

Two-sample scale problem deals with testing equality of variabilities of two absolutely continuous populations. Let $X_1, X_2, \cdots, X_m$ be a random sample of size m from $F(x)$ and $Y_1, Y_2, \cdots, Y_n$ be a random sample of size n from $G(x) = F\left(\frac{x}{\sigma}\right), \sigma > 0$. Here $F(x)$ and $G(x)$ are absolutely continuous distribution functions. We consider testing

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2010 AMS Subject Classification : 62G10, 62G20.

$H_0 : \sigma = 1$ vs $H_1 : \sigma > 1$, that is $H_0 : F(x) = G(x)$ vs $H_1 : F(x) > G(x)$.

The two-sample scale problem is a fundamental problem that occurs in all walks of life. For example, comparing variances of amounts of milk filled in cans of 500ml by two machines [3], comparing peak levels of human plasma growth hormone in 10 relatively coronary-prone subjects and in 11 relatively coronary-resistant subjects after infusing the subjects with arginine hydrochloride [10], comparing variability in time to breakdown of an insulating fluid under different voltages [11] and comparing variability of decisions made by novice and experienced inspectors on a finished product [20].

An extensive study has been done on the two-sample scale problem. Some early works are due to Lehmann [8], Rosenbaum [15], Mood [13], Sukhatme [21, 22] and Siegel and Tukey [16]. Duran [4] carried out a substantial review on this problem. Kusum [5] modified the test proposed by Deshpande and Kusum [3]. Shetty and Bhat [17], Shetty and Pandit [18], Mahajan et.al. [9] and Bhat et. al. [2] proposed outlier resistant tests for the problem. Mehra and Rao [12], Kochar and Gupta [6] and Kossler and Narinder Kumar [7] studied the tests based on extreme order statistics. Since such observations contain the information about the tails of the distributions, tests based them are useful in exploring the two-sample scale problem.

We propose two classes of tests based on extreme observations. The classes of tests constitute U-Statistics with kernels as functions of extreme order statistics of subsamples of size b from X-sample and of size d from Y-sample.

In section 2, we suggest two classes of tests and furnish their alternative forms in terms of ordered ranks. Section 3 deals with their distributions. In section 4, we present their performances in terms of ARE and empirical power. We give concluding remarks about the proposed classes of tests along with their applications in section 5.

2. Suggested Classes of Tests

In this section, we suggest two classes of tests $C_1(b, d)$ and $C_2(b, d)$ based on extreme order statistics of subsamples of sizes b and d and provide their alternative forms in terms of ordered ranks assuming that there are no ties. The tests are based on two-sample U-statistics depending on the kernels being functions of subsample extrema.

Defining,

$$\varphi_1(x_1,x_2,\cdots,x_b;y_1,y_2,\cdots,y_d)=\begin{cases}1 & \text{if } 0<x^+_{(b)}<y^+_{(d)}, x_i,y_j>0\\ & \text{or } y^-_{(1)}<x^-_{(1)}<0, x_i,y_j<0,\\ 0 & \text{Otherwise}\end{cases} \quad (1)$$

$$\varphi_2(x_1,x_2,\cdots,x_b;y_1,y_2,\cdots,y_d)=\begin{cases}1 & \text{if } 0<x^+_{(1)}<y^+_{(1)}, x_i,y_j>0\\ & \text{or } y^-_{(d)}<x^-_{(b)}<0, x_i,y_j<0,\\ 0 & \text{Otherwise}\end{cases} \quad (2)$$

where $x^+_{(b)}=Max(x^+_1,\cdots,x^+_b), y^+_{(d)}=Max(y^+_1,\cdots,y^+_d)$, $x^+_{(1)}=Min(x^+_1,\cdots,x^+_b)$, $y^+_{(1)}=Min(y^+_1,\cdots,y^+_d)$, $x^-_{(b)}=Max(x^-_1,\cdots,m^-_b)$, $y^-_{(d)}=Max(y^-_1,\cdots,y^-_d)$, $x^-_{(1)}=Min(x^-_1,\cdots,x^-_b), y^-_{(1)}=Min(y^-_1,\cdots,y^-_d)$, $Max(\cdot)$ is maximum, $Min(\cdot)$ is minimum, $x^+_i(x^-_i)i=1,2,\cdots,b$ are positive (negative) X observations and $y^+_j(y^-_j)$ $j=1,2,\cdots,d$ are positive (negative) Y observations, we propose

$$C_1(b,d)=\left(\begin{pmatrix}m\\b\end{pmatrix}\begin{pmatrix}n\\d\end{pmatrix}\right)^{-1}\sum_{\mathcal{C}}\varphi_1(X_{i_1},X_{i_2},\cdots,X_{i_b};Y_{j_1},Y_{j_2},\cdots,Y_{j_d}) \quad (3)$$

and

$$C_2(b,d)=\left(\begin{pmatrix}m\\b\end{pmatrix}\begin{pmatrix}n\\d\end{pmatrix}\right)^{-1}\sum_{\mathcal{C}}\varphi_2(X_{i_1},X_{i_2},\cdots,X_{i_b};Y_{j_1},Y_{j_2},\cdots,Y_{j_d}) \quad (4)$$

where $\mathcal{C}$ denotes sum over all possible $\begin{pmatrix}m\\b\end{pmatrix}\begin{pmatrix}n\\d\end{pmatrix}$ combinations of X and Y-sample observations.

$C_1(b,d)$ and $C_2(b,d)$ are distribution-free under H_0 for all values of m,n for $1\le b\le m$, $1\le d\le n$ and their large values are significant for testing H_0 against H_1.

Both the proposed classes of tests include Sukhatme [21] test as their particular case for $b=d=1$. Test studied by Mehra and Rao [12] and Kossler and Narinder Kumar [7] is a particular case of $C_1(b,d)$ for $b=d$ when the alternative hypothesis is one sided and the distributions are having common quantile as median.

Suppose $m=m^++m^-$ $(n=n^++n^-)$, $X^+_{(1)}<X^+_{(2)}<\cdots<X^+_{m^+}$ $(Y^+_{(1)}<Y^+_{(2)}<\cdots,<Y^+_{n^+})$ are ordered positive observations and $X^-_{(1)}<X^-_{(2)}<\cdots<X^-_{m^-}$ $(Y^-_{(1)}<Y^-_{(2)}<\cdots<Y^-_{n^-})$ are ordered negative observations from $X(Y)$-sample, $R^+_{(i)}(R^-_{(i)})$ represents the rank of $X^+_{(i)}(X^-_{(i)})$ and $S^+_{(j)}(S^-_{(j)})$ represents the rank of $Y^+_{(j)}(Y^-_{(j)})$ in the joint rankings

of $X_1^+,\cdots,X_{m^+}^+,Y_1^+,\cdots,Y_{n+}$, $(X_1^-.\cdots,X_{m^-}^-,Y_1^-,\cdots,Y_{n^-}^-)$. Following Bhat [1] and Shetty et. al. [19], the alternative expressions of $C_1^*(b,d)=\binom{m}{b}\binom{n}{d}C_1(b,d)$ and $C_2^*(b,d)=\binom{m}{b}\binom{n}{d}C_2(b,d)$ are given in the form of ordered ranks as follows.

$$\begin{aligned} C_1^*(b,d) &= \sum_{i=1}^{m^+}\sum_{l=0}^{d-1}\binom{i-1}{b-1}\binom{R^+_{(i)}-i}{d-l-1}\binom{n^+R^+_{(i)}+i}{l+1} \\ &\quad+\sum_{j=1}^{n^-}\binom{n^--j}{d-1}\binom{m^--S^-_{(j)}+j}{b} \end{aligned} \tag{5}$$

and

$$\begin{aligned} C_2^*(b,d) &= \sum_{i=1}^{m^+}\binom{m^+-i}{b-1}\binom{n^+-R^+_{(i)}+i}{d} \\ &\quad+\sum_{j=1}^{n^-}\sum_{l=0}^{b-1}\binom{j-1}{d-1}\binom{S^-_{(j)}-j}{b-l-1}\binom{m^--S^-_{(j)}+j}{l+1} \end{aligned} \tag{6}$$

3. Distribution of $C_1(b,d)$ and $C_2(b,d)$

In this section, we derive the null mean and asymptotic distributions of $C_1(b,d)$ and $C_2(b,d)$. Also, we obtain the null distribution of $C_1^*(b,d)$ and $C_2^*(b,d)$.

The mean of $C_1(b,d)$ is given by

$$\begin{aligned} E[C_1(b,d)] &= P[0<X_{(b)}<Y_{(d)}]+P[Y_{(1)}<X_{(1)}<0] \\ &= d\Bigg[\int_0^\infty (2F(x)-1)^b(2G(x)-1)^{d-1}2dG(x) \\ &\quad+\int_{-\infty}^0 (1-2F(x))^b(1-2G(x))^{d-1}2dG(x)\Bigg]. \end{aligned}$$

Under H_0, the null mean of $C_1(b,d)$ is given by

$$\mu_{0_1}=E_{H_0}[C_1(b,d)]=\frac{2d}{b+d}. \tag{7}$$

The mean of $C_2(b,d)$ is

$$E[C_2(b,d)]=P[0<X_{(1)}<Y_{(1)}]+P[Y_{(d)}<X_{(b)}<0]$$

and under H_0, the null mean of $C_2(b,d)$ is

$$\mu_{0_2}=E_{H_0}[C_2(b,d)]=\frac{2b}{b+d}. \tag{8}$$

When $b = d, \mu_{0_1} = \mu_{0_2} = 1$.

The asymptotic null variance of $C_1(b,d)$ is given by

$$\sigma_1^2 = \frac{b^2(\xi_{10})_{C_1}}{\lambda} + \frac{d^2(\xi_{01})_{C_1}}{1-\lambda} \tag{9}$$

where $0 < \lambda = \lim_{N\to\infty} \frac{m}{N} < 1, N = m+n$,

$$\begin{aligned}
(\xi_{10})_{C_1} &= Cov[\phi_1(X_1,\cdots,X_b;Y_1,\cdots,Y_d),\\
&\quad \phi_1(X_1,X_{b+1},\cdots,X_{2b-1};Y_{d+1},\cdots,Y_{2d}]\\
&= \int_{-\infty}^{\infty} P^2[(0 < Max(x,X_2,\cdots,X_b) < Max(Y_1,\cdots,Y_d))\\
&\quad +(Min(Y_1,\cdots,Y_d) < Min(x,X_2,\cdots,X_b) < 0]2dF(x) - \mu_{0_1}^2\\
&= \frac{2d^2}{(b+d-1)^2}\left\{\frac{b+d-2}{b+d} + \frac{1}{2b+2d-1}\right\} - \frac{2d^2}{(b+d)^2}\\
&= \frac{2d^2}{(b+d-1)^2}\left\{\frac{1}{2b+2d-1} - \frac{1}{(b+d)^2}\right\} \qquad (10)
\end{aligned}$$

and

$$\begin{aligned}
(\xi_{01})_{C_1} &= Cov[\phi_1(X_1,\cdots,X_b;Y_1,\cdots,Y_d),\\
&\quad \phi_1(X_{b+1},\cdots,X_{2b};Y_1,Y_{d+1},\cdots,Y_{2d-1}]\\
&= \int_{-\infty}^{\infty} P^2[(0 < Max(X_1,\cdots,X_b) < Max(y,Y_2,\cdots,Y_d))\\
&\quad +(Min(y,Y_2,\cdots,Y_d) < Min(X_1,\cdots,X_b) < 0]2dF(x) - \mu_{0_1}^2\\
&= \frac{2b^2}{(b+d-1)^2}\left\{\frac{1}{2b+2d-1} - \frac{1}{(b+d)^2}\right\} \qquad (11)
\end{aligned}$$

Since $b^2(\xi_{10})_{C_1} = d^2(\xi_{01})_{C_1}$, we have

$$\sigma_1^2 = \frac{b^2(\xi_{10})_{C_1}}{\lambda(1-\lambda)}. \tag{12}$$

For $b = d$,

$$(\xi_{10}^*)_{C_1} = \frac{1}{2(4b-1)}, \quad \sigma_1^{*^2} = \frac{b^2}{2\lambda(1-\lambda)(4b-1)}. \tag{13}$$

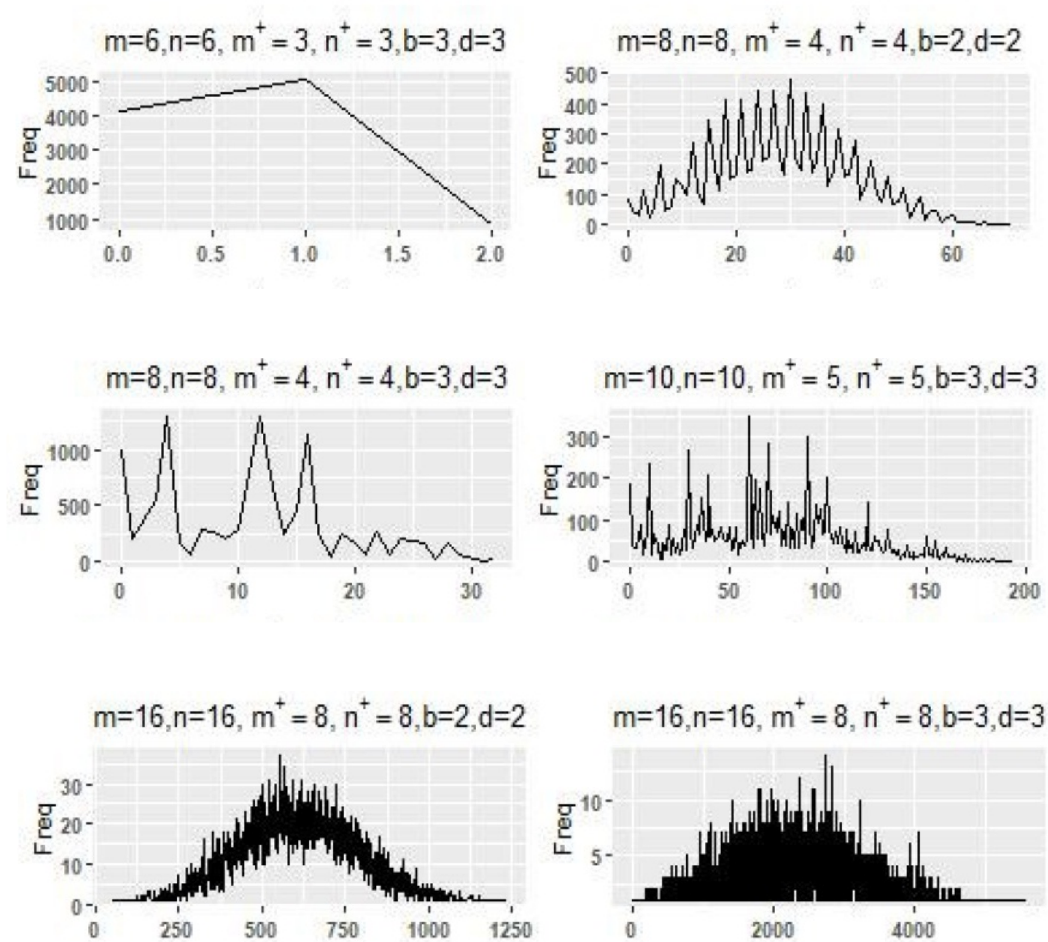
By generalized U-statistic theorem due to Lehmann [8], $\sqrt{N}C_1(b,d)$ follows asymptotic normal distribution with mean μ_{0_1} and variance σ_1^2 and $\sqrt{N}C_1(b,b)$ follows asymptotic normal distribution with unit mean and variance $\sigma_1^{*^2}$.

Proceeding on similar grounds, the asymptotic distribution of $\sqrt{N}C_2(b,d)$ has asymptotic normal distribution with mean $\mu_{0_2} = \frac{2b}{b+d}$ and variance $\sigma_2^2 = \frac{b^2(\xi_{10})_{C_2}}{\lambda(1-\lambda)}$, where

$$\begin{aligned}(\xi_{10})_{C_2} &= \frac{2}{(b+d-1)^2}\left\{(b-1)^2 + \frac{d^2}{2b+2d-1} + \frac{2d(b-1)}{b+d}\right\} - \frac{2b^2}{(b+d)^2} \\ &= \frac{2d^2}{(b+d-1)^2}\left\{\frac{1}{2b+2d-1} - \frac{1}{(b+d)^2}\right\}.\end{aligned} \tag{14}$$

By (10) and (14), we get, $\sigma_1^2 = \sigma_2^2$.

We observe that, the numerator of mean of $C_1(b,d)$ depends on subsample size of Y sample whereas that of $C_2(b,d)$ depends on subsample size of X sample and both the proposed classes of tests have equal variances.



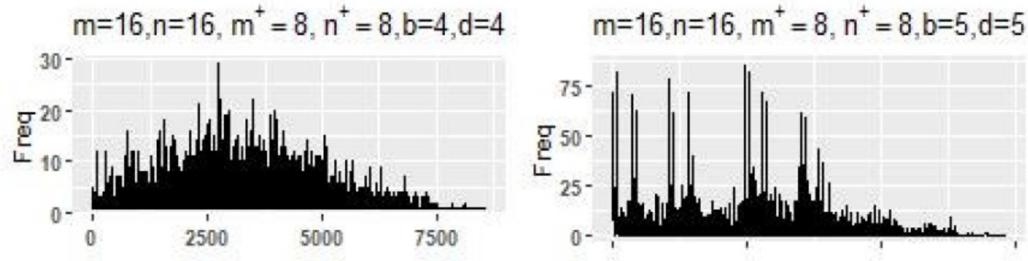


Fig 1: Null distribution of $C_1^*(b,d)$.

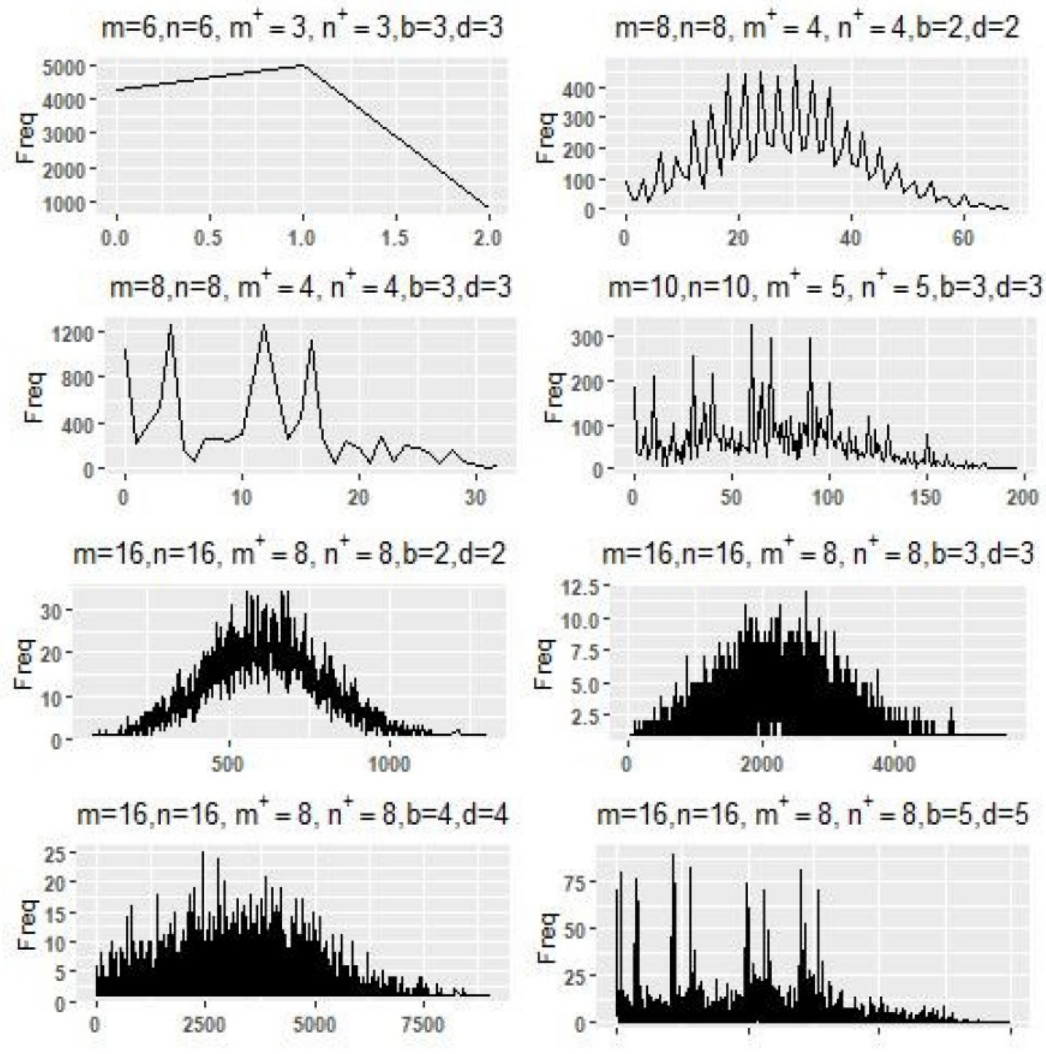


Fig 2: Null distribution of $C_2^*(b,d)$.

To obtain null distribution of $C_1^*(b,d)$ and $C_2^*(b,d)$, 10000 random samples are generated from uniform distribution for specified values of m, n, m^+, n^+, b and d. The null distributions of the proposed classes of tests are obtained using Monte-Carlo simulation technique. Figure 1 and figure 2 respectively exhibit the null distributions of $C_1^*(b,d)$ and $C_2^*(b,d)$ for $m = n, m^+ = n^+$ and $b = d$.
From the figures we observe that, the null distributions of both the classes of tests are normal for higher values of m, n, m^+, n^+ and for relatively smaller values of b, d. The null distributions are skewed when b and d are relatively large.

4. Performance of $C_1(b,d)$ and $C_2(b,d)$

In this section, we discuss about the large sample performance of $C_1(b,d)$ and $C_2(b,d)$ in terms of Pitman ARE and their small sample performance in terms of empirical power. The efficacies of $C_1(b,d)$ and $C_2(b,d)$ are given by

$$e(C_1(b,d)) = \frac{\left[\frac{d}{d\sigma}[E_{H_1}(C_1(b,d))]_{\sigma=1}\right]^2}{\sigma_1^2} = \frac{b^2d^2[I_{1C_1} - I_{2C_1}]^2}{\sigma_1^2} \tag{15}$$

and

$$e(C_2(b,d)) = \frac{\left[\frac{d}{d\sigma}[E_{H_1}(C_2(b,d))]_{\sigma=1}\right]^2}{\sigma_2^2} = \frac{b^2d^2[I_{1C_2} - I_{2C_2}]^2}{\sigma_2^2} \tag{16}$$

where

$$\begin{aligned}
I_{1C_1} &= 4\int_0^\infty x(2F(x)-1)^{b+d-2} f^2(x)dx, \\
I_{2C_1} &= 4\int_{-\infty}^0 x(1-2F(x))^{b+d-2} f^2(x)dx. \\
I_{1C_2} &= 4\int_0^\infty x(2\overline{F}(x))^{b+d-2} f^2(x)dx \quad \text{and} \\
I_{2C_2} &= 4\int_{-\infty}^0 x(2F(x))^{b+d-2} f^2(x)dx.
\end{aligned}$$

The ARE of $C_1(b,d)$ and $C_2(b,d)$ with respect to (wrt) any other test T is given by

$$ARE(C_1(b,d),T) = \frac{e(C_1(b,d))}{e(T)} \quad \text{and} \quad ARE(C_2(b,d),T) = \frac{e(C_2(b,d))}{e(T)}. \tag{17}$$

We compare the asymptotic performances of $C_1(b,d)$ and $C_2(b,d)$ wrt M [13], ST [16], T_1 [3], T_2 [5], $U_1(a_1,a_2)$ [6], $U_2(3,k)$[17] and $U_3(c_1,c_2)$ [2].

The efficacy values of $C_1(b,d)$, $C_2(b,d)$ for various distributions and values of $b+d$ are computed in table 1 of appendix. Since these values are equal for $b+d$ for varying values of b,d, we furnish the efficacy values and compute ARE for sum of the subsample sizes $s=b+d$. Also we take $a=a_1+a_2$ and $c=c_1+c_2$. The ARE of the proposed classes of tests wrt M, ST are given in table 2, wrt T_1, T_2 in table 3, wrt $U_1(a_1,a_2), U_2(3,k)$ and $U_3(c_1,c_2)$ are given respectively in tables 4, 5 and 6 in appendix. The critical values with attained level of significance (α^*) for computing empirical powers of $C_1^*(b,d)$ and $C_2^*(b,d)$ are given in table 7. And the empirical powers of $C_1^*(b,d)$ and $C_2^*(b,d)$ for 10% level of significance ($\alpha=0.10$) under various distributions are respectively furnished in table 8 and table 9.

From table 1, we observe that, the efficacies of $C_1(b,d)$ and $C_2(b,d)$ are equal for Cauchy distribution, but efficacies of $C_1(b,d)$ are higher than those of $C_2(b,d)$ for all other distributions under consideration. Tables 2, 4, 5 and 6 reveal that $C_1(b,d)$ outperforms $C_2(b,d)$ wrt their large sample performances with other tests under consideration.

Table 2 shows that, $C_1(b,d)$ outperforms M and ST under uniform, normal, logistic, Laplace distributions for all values of s and under Cauchy distribution for $s \leq 8$. The ARE of $C_1(b,d)$ wrt M and ST is increasing for increasing values of s for distributions other than Cauchy distribution. The performance of $C_2(b,d)$ is better than M and ST under normal distribution for smaller values of b,d, under logistic and Laplace distribution for $s<5$ and it decreases with increasing values of s.

Table 3 reveals that, under normal distribution both the proposed classes of tests outperform T_1, T_2 and the ARE values of $C_1(b,d)$ increase with increasing values of s whereas, they decrease in case of $C_2(b,d)$. Under Laplace distribution, $C_1(b,d)$ outperforms T_1, T_2 and ARE values decrease as s increases. $C_2(b,d)$ performs better than T_1 and T_2 for $s<4$. In case of Cauchy distribution, both the proposed classes of tests are better than T_1 and T_2 respectively for $s \leq 8$ and $s \leq 9$.

Table 4 shows that, $C_1(b,d)$ outperforms $U_1(a_1,a_2)$ under exponential, normal, logistic and Laplace distributions whereas, $C_2(b,d)$ outperforms $U_1(a_1,a_2)$ under normal distribution, under exponential distribution and logistic distribution respectively for $s<7$ and $s<5$. The ARE of $C_1(b,d)$ wrt $U_1(a_1,a_2)$ is increasing with increasing s and decreasing for increasing values of a for a given s.

It can be seen from table 5 that, $C_1(b,d)$ outperforms $U_2(3,k)$. The ARE is increasing for increasing s and is decreasing for a given value of s when k increases. $C_2(b,d)$ is better than $U_2(3,k)$ under normal distribution for all values of s and under logistic and Laplace distribution for $s \leq 4$.

Table 6 shows that, $C_1(b,d)$ outperforms $U_3(c_1,c_2)$ under all the distributions considered whereas, $C_2(b,d)$ is better than $U_3(c_1,c_2)$ under normal distribution. The ARE of $C_1(b,d)$ is increasing for increasing values of s and c whereas, the ARE $C_2(b,d)$ is decreasing with increasing values of s.

With respect to small sample performance, from table 8, we find that for $\sigma \leq 1.2$, the empirical power of $C_1^*(b,d)$ is higher under normal, logistic, Laplace and Cauchy distributions than it's empirical power under uniform distribution. For $\sigma > 1.2$, the empirical power under normal and uniform distributions are higher. The empirical power is higher for larger m, n, m^+, n^+ and smaller b, d.

We observe from table 9 that, the empirical power of $C_2^*(b,d)$ is higher for Cauchy distribution when compared to other distributions under consideration. The empirical power is higher for smaller values of b, d and larger values of m, n, m^+, n^+.

5. Conclusion

In this section, we consider an example for illustrating the application of proposed classes of tests.

Example: **[14]** Let X and Y respectively represent the times in minutes to breakdown of an insulating fluid under elevated voltage stress of 32kV and 36kV.

X : 0.27, 0.4, 0.69, 0.79, 2.75, 3.91, 9.88, 13.95, 15.93, 27.8, 53.24, 82.85, 89.29, 100.58, 215.5.

Y : 0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.9, 3.67, 3.99, 5.35, 13.77, 25.5.

In testing for possible difference of variances among the two samples the p-values of the proposed classes of tests are given below.

Test	$C_1(2,2)$	$C_1(3,3)$	$C_1(4,4)$	$C_2(2,2)$	$C_2(3,3)$	$C_2(4,4)$	M	F-test
p-value	0.0000	0.0001	0.0021	0.0001	0.0005	0.0016	0.0162	0.0000

Here, we observe that the p-values of both the proposed classes of tests are smaller than

that of M-test. Also, the classes of tests have p-value almost similar to that of F-test when the subsample sizes are small. Hence the suggested classes of tests are effective in testing equality of variability in two populations.
We conclude that,

- The two-classes of suggested tests based on U-statistics being functions of subsample extrema for two-sample scale problem are distribution-free and their large values are significant for testing H_0 vs H_1.
- Both the classes of tests have equal asymptotic variances and they follow asymptotic normal distribution.
- The class of tests $C_1(b, d)$ outperforms $C_2(b, d)$ in terms of Pitman ARE.
- The classes of tests $C_1(b, d)$ and $C_2(b, d)$ outperform all the competitors respectively for distributions under consideration and normal distribution.
- The class of tests $C_2(b, d)$ is better than M, ST for $s \leq 8$, is better than T_1 for $s \leq 8$ and is better than T_2 for $s \leq 9$ under Cauchy distribution. It is better than $U_1(a_1, a_2)$ under exponential and logistic distributions respectively for s and $a \leq 6$ and $s \leq 4$ and $a \leq 6$. Also, the class of tests is better than $U_2(3, k)$ under logistic and Laplace distribution respectively for $s \leq 4, k \leq 5$ and $s \leq 4, 5 \leq k \leq 7$.
- The class of tests $C_1^*(b, d)$ has higher empirical power than $C_2^*(b, d)$.
- Empirical power of $C_1^*(b, d)$ is higher for light and medium tailed distributions whereas, that of $C_2^*(b, d)$ is higher for heavy tailed distributions.
- For smaller values of b, d both the classes of tests have similar p-value as that of F-test and their p-values are smaller than that of Mood's test.
- Although the asymptotic variances of $C_1(b, d)$ and $C_2(b, d)$ are equal, $C_1(b, d)$ outperforms $C_2(b, d)$ in terms of Pitman ARE and empirical power.

Appendix

Table 1: Efficacies of $C_1(b,d)$ and $C_2(b,d)$.

	s	Uniform	Triangular	Exponential	Normal	Logistic	Laplace	Cauchy
$C_1(b,d)$	4	14.0000	4.2720	0.4560	53.7980	2.6660	1.8260	0.8130
	5	18.0000	4.7900	0.4630	56.6310	2.7220	1.8530	0.7100
	6	22.0000	5.2090	0.4630	58.1570	2.7250	1.8500	0.6230
	7	26.0000	5.5570	0.4580	58.9020	2.7000	1.8320	.5520
	8	30.0000	5.8540	0.4520	59.1570	2.6590	1.8070	0.4940
	9	34.0000	6.1110	0.4440	59.0980	2.6110	1.7770	0.4450
	10	38.0000	6.3380	0.4360	58.8320	2.5600	1.7460	0.4050
$C_2(b,d)$	4	1.5560	1.1430	0.2190	19.2430	1.1350	0.8750	0.8130
	5	1.1250	0.8890	0.1800	15.1330	0.9080	0.7200	0.7100
	6	0.8800	0.7270	0.1530	12.4190	0.7530	0.6110	0.6230
	7	0.7220	0.6150	0.1330	10.5050	0.6410	0.5310	0.5520
	8	0.6120	0.5330	0.1170	9.0880	0.5570	0.4690	0.4940
	9	0.5310	0.4710	0.1050	8.0000	0.4920	0.4200	0.4450
	10	0.4690	0.4210	0.0950	7.1400	0.4400	0.3800	0.4050

Table 2: ARE of $C_1(b,d)$ and $C_2(b,d)$ wrt to M and ST.

		Uniform		Normal		Logistic		Laplace		Cauchy	
	s	M	ST	M	ST	M	ST	M	ST	M	ST
$C_1(b,d)$	4	2.800	4.667	35.398	44.249	2.860	2.546	2.102	2.444	1.764	1.651
	5	3.600	6.000	37.262	46.580	2.921	2.601	2.133	2.480	1.540	1.442
	6	4.400	7.334	38.267	47.835	2.924	2.603	2.130	2.477	1.352	1.266
	7	5.200	8.667	38.756	48.447	2.897	2.579	2.109	2.453	1.198	1.121
	8	6.000	10.001	38.925	48.657	2.854	2.540	2.080	2.418	1.071	1.003
	9	6.800	11.334	38.886	48.609	2.802	2.495	2.046	2.379	0.966	0.905
	10	7.600	12.668	38.711	48.390	2.747	2.446	2.009	2.337	0.878	0.822
$C_2(b,d)$	4	0.311	0.519	12.662	15.828	1.218	1.084	1.007	1.171	1.764	1.651
	5	0.225	0.375	9.958	12.447	0.974	0.867	0.829	0.964	1.540	1.442
	6	0.176	0.293	8.172	10.215	0.808	0.719	0.704	0.818	1.352	1.266
	7	0.144	0.241	6.912	8.640	0.688	0.612	0.611	0.710	1.198	1.121
	8	0.122	0.204	5.980	7.475	0.598	0.532	0.540	0.627	1.071	1.003
	9	0.106	0.177	5.264	6.580	0.528	0.470	0.483	0.562	0.966	0.905
	10	0.094	0.156	4.698	5.873	0.473	0.421	0.437	0.509	0.878	0.822

Table 3 : ARE of $C_1(b,d)$ and $C_2(b,d)$ wrt to T_1 and T_2.

		T_1			T_2		
	s	Normal	Laplace	Cauchy	Normal	Laplace	Cauchy
$C_1(,d)$	4	44.2817	2.4322	1.6534	32.0263	1.9982	2.0012
	5	46.6140	2.4684	1.4436	33.7131	2.0280	1.7473
	6	47.8701	2.4650	1.2675	34.6216	2.0252	1.5342
	7	48.4828	2.4412	1.1228	35.0647	2.0057	1.3590
	8	48.6932	2.4071	1.0039	35.2169	1.9777	1.2151
	9	48.6446	2.3676	0.9057	35.1817	1.9452	1.0962
	10	48.4258	2.3256	0.8233	35.0235	1.9107	0.9966
$C_2(b,d)$	4	15.8393	1.1657	1.6534	11.4556	0.9577	2.0012
	5	12.4565	0.9592	1.4436	9.0090	0.7881	1.7473
	6	10.2222	0.8141	1.2675	7.3931	0.6689	1.5342
	7	8.6467	0.7069	1.1228	6.2537	0.5808	1.3590
	8	7.4806	0.6244	1.0039	5.4103	0.5130	1.2151
	9	6.5851	0.5593	0.9057	4.7626	0.4595	1.0962
	10	5.8769	0.5063	0.8233	4.2504	0.4159	0.9966

Table 4 : ARE of $C_1(b,d)$ and $C_2(b,d)$ wrt $U_1(a_1,a_2)$

		Exponential		Normal		Logistic		Laplace	
s	a	$C_1(b,d)$	$C_2(b,d)$	$C_1(b,d)$	$C_2(b,d)$	$C_1(b,d)$	$C_2(b,d)$	$C_1(b,d)$	$C_2(b,d)$
4	5	3.1939	1.5308	34.3569	12.2893	2.8018	1.1931	1.6184	0.7757
	6	3.0859	1.4790	33.1286	11.8500	2.7353	1.1648	1.3997	0.6709
5	5	3.2414	1.2596	36.1665	9.6647	2.8614	0.9541	1.6426	0.6383
	6	3.1319	1.2170	34.8735	9.3192	2.7935	0.9315	1.4206	0.5520
6	5	3.2369	1.0691	37.1411	7.9312	2.8642	0.7911	1.6402	0.5418
	6	3.1274	1.0330	35.8133	7.6476	2.7962	0.7724	1.4186	0.4685
7	5	3.2058	0.9283	37.6164	6.7087	2.8375	0.6738	1.6245	0.4704
	6	3.0974	0.8692	36.2716	6.4689	2.7702	0.6578	1.4049	0.4068

Table 5 : ARE of $C_1(b, d)$ and $C_2(b, d)$ wrt $U_2(3, k)$

		$C_1, (b, d)$				$C_2, (b, d)$			
s	k	Uniform	Normal	Logistic	Laplace	Uniform	Normal	Logistic	Laplace
	2	5.9998	45.3240	2.5720	1.2244	0.6668	16.2119	1.0950	0.5867
	5	5.0976	42.0380	2.4203	2.2594	0.5666	15.0366	1.0304	1.0827
4	7	4.1408	38.4708	2.2618	2.2133	0.4602	13.7606	0.9629	1.0606
	2	7.7141	47.7107	2.6260	1.2425	0.4821	12.7493	0.8760	0.4828
	5	6.5540	44.2517	2.4712	2.2928	0.4096	11.8250	0.8243	0.8909
5	7	5.3239	40.4967	2.3093	2.2460	0.3327	10.8216	0.7703	0.8727
	2	9.4283	48.9963	2.6289	1.2405	0.3771	10.4628	0.7264	0.4097
	5	8.0105	45.4441	2.4739	2.2891	0.3204	9.7043	0.6836	0.7560
6	7	6.5070	41.5879	2.3118	2.2424	0.2603	8.8808	0.6388	0.7406
	2	11.1425	49.6240	2.6048	1.2284	0.3094	8.8503	0.6184	0.3560
	5	9.4669	46.0263	2.4512	2.2668	0.2629	8.2087	0.5819	0.6570
	7	7.6900	42.1207	2.2906	2.2206	0.2135	7.5121	0.5438	0.6436
	2	12.8568	49.8388	2.5652	1.2116	0.2623	7.6565	0.5374	0.3145
	5	10.9234	46.2255	2.4140	2.2359	0.2228	7.1014	0.5057	0.5803
8	7	8.8731	42.3030	2.2558	2.1903	0.1810	6.4988	0.4725	0.5685
	2	14.5710	49.7891	2.5189	1.1915	0.2276	6.7399	0.4746	0.2816
	5	12.3798	46.1794	2.3704	2.1988	0.1933	6.2512	0.4467	0.5197
9	7	10.0562	42.2608	2.2151	2.1539	0.1571	5.7208	0.4174	0.5091
	2	16.2852	49.5650	2.4697	1.1707	0.2010	6.0153	0.4245	0.2548
	5	13.8363	45.9716	2.3241	2.1604	0.1708	5.5792	0.3995	0.4702
10	7	11.2393	42.0706	2.1718	2.1163	0.1387	5.1058	0.3733	0.4606

Table 6 : ARE of $C_1(b,d)$ and $C_2(b,d)$ wrt $U_3(c_1,c_2)$

	s	c	Uniform	Triangular	Exponential	Normal	Logistic	Laplace	Cauchy
$C_1(b,d)$	4	6	3.9697	2.0401	1.3600	25.5104	1.4139	1.3615	0.8150
		8	4.3349	2.1478	1.4027	26.4634	1.4566	1.4043	0.8199
		10	4.5940	2.2270	1.4359	27.1904	1.4899	1.4374	0.8254
	5	6	5.1039	2.2874	1.3809	26.8538	1.4436	1.3817	0.7118
		8	5.5734	2.4082	1.4243	27.8570	1.4872	1.4250	0.7160
		10	5.9065	2.4970	1.4579	28.6222	1.5212	1.4587	0.7208
	6	6	6.2381	2.4875	1.3809	27.5774	1.4452	1.3794	0.6246
		8	6.8120	2.6189	1.4243	28.6076	1.4888	1.4227	0.6283
		10	7.2191	2.7154	1.4579	29.3935	1.5229	1.4563	0.6325
	7	6	7.3723	2.6537	1.3660	27.9307	1.4319	1.3660	0.5534
		8	8.0505	2.7939	1.4089	28.9741	1.4752	1.4089	0.5567
		10	8.5317	2.8968	1.4422	29.7700	1.5089	1.4422	0.5604
	8	6	8.5065	2.7955	1.3481	28.0516	1.4102	1.3474	0.4952
		8	9.2890	2.9432	1.3904	29.0995	1.4528	1.3897	0.4982
		10	9.8442	3.0517	1.4233	29.8989	1.4860	1.4225	0.5015
	9	6	9.6407	2.9183	1.3243	28.0236	1.3847	1.3250	0.4461
		8	10.5276	3.0724	1.3658	29.0705	1.4265	1.3666	0.4488
		10	11.1568	3.1856	1.3981	29.8691	1.4592	1.3989	0.4518
	10	6	10.7749	3.0267	1.3004	27.8975	1.3577	1.3019	0.4060
		8	11.7661	3.1865	1.3412	28.9397	1.3987	1.3428	0.4084
		10	12.4694	3.3040	1.3729	29.7347	1.4307	1.3745	0.4112
$C_2(b,d)$	4	6	0.4412	0.5458	0.6532	9.1248	0.6019	0.6524	0.8150
		8	0.4818	0.5747	0.6737	9.4657	0.6201	0.6729	0.8199
		10	0.5106	0.5958	0.6896	9.7257	0.6343	0.6888	0.8254
	5	6	0.3190	0.4245	0.5369	7.1759	0.4815	0.5369	0.7118
		8	0.3483	0.4470	0.5537	7.4440	0.4961	0.5537	0.7160
		10	0.3692	0.4634	0.5668	7.6485	0.5074	0.5668	0.7208
	6	6	0.2495	0.3472	0.4563	5.8890	0.3993	0.4556	0.6246
		8	0.2725	0.3655	0.4707	6.1090	0.4114	0.4699	0.6283
		10	0.2888	0.3790	0.4818	6.2768	0.4208	0.4810	0.6325
	7	6	0.2047	0.2937	0.3967	4.9814	0.3399	0.3959	0.5534
		8	0.2236	0.3092	0.4091	5.1674	0.3502	0.4084	0.5567
		10	0.2369	0.3206	0.4188	5.3094	0.3582	0.4180	0.5604
	8	6	0.1735	0.2545	0.3490	4.3094	0.2954	0.3497	0.4952
		8	0.1895	0.2680	0.3599	4.4704	0.3043	0.3607	0.4982
		10	0.2008	0.2779	0.3684	4.5932	0.3113	0.3692	0.5015
	9	6	0.1506	0.2249	0.3132	3.7935	0.2609	0.3132	0.4461
		8	0.1644	0.2368	0.3230	3.9352	0.2688	0.3230	0.4488
		10	0.1742	0.2455	0.3306	4.0433	0.2750	0.3306	0.4518
	10	6	0.1330	0.2010	0.2833	3.3857	0.2333	0.2833	0.4060
		8	0.1452	0.2117	0.2922	3.5122	0.2404	0.2922	0.4084
		10	0.1539	0.2195	0.2991	3.6087	0.2459	0.2991	0.4112

Table 7 : Critical values and α^* for $C_1^*(b,d)$ and $C_2^*(b,d)$ for different values of m, n, m^+, n^+, b and d

						Critical Values (α^*)			
m	n	m^+	n^+	b	d	$C_1^*(b,d)$	$C_2^*(b,d)$	$C_1^*(b,d)$	$C_2^*(b,d)$
6	6	2	2	2	2	30 (0.0414)	31 (0.0385)	27 (0.0892)	39 (0.0985)
6	6	3	3	2	2	14 (0.0474)	15 (0.0152)	12 (0.0913)	12 (0.0911)
6	6	3	3	3	2	-	5 (0.0376)	5 (0.0961)	4 (0.0801)
7	6	4	3	3	3	4 (0.0284)	-	-	4 (0.0802)
7	9	4	4	3	3	25 (0.0468)	19 (0.0322)	22 (0.0957)	16 (0.0874
8	8	2	2	2	2	163 (0.0470)	165 (0.0477)	147 (0.0970)	147 (0.0989)
8	8	3	3	3	3	73 (0.0499)	93 (0.0482)	65 (0.0963)	90 (0.0643)
8	8	4	4	2	2	50 (0.0480)	51 (0.0425)	45 (0.0946)	46 (0.0933)
8	8	4	4	2	3	34 (0.0477)	41 (0.0484)	30 (0.0941)	38 (0.0772)
8	8	4	4	3	2	41 (0.0451)	34 (0.0455)	37 (0.0979)	30 (0.0887)
10	5	5	2	2	130	132 (0.0484)	118 (0.0469)	120 (0.0984)	(0.0963)
10	10	5	5	2	3	131 (0.0486)	155 (0.0485)	118 (0.0980)	120 (0.0962)
10	10	5	5	3	2	155 (0.0485)	130 (0.0478)	140 (0.0946)	118 0.0958)
10	10	5	5	2	4	70 (0.0456)	115 (0.0464)	61 (0.0966)	101 (0.0993)
10	10	5	5	3	4	70 (0.0380)	85 (0.0416)	60 (0.0930)	75 (0.0971)
10	10	5	5	4	4	40 (0.0290)	40 (0.0298)	31 (0.0977)	31 (0.0986)
16	16	8	8	2	2	912 (0.0499)	906 (0.0949)	848 (0.0995)	841 (0.0996)
16	16	8	8	3	3	3817 (0.0499)	3821 (0.0499)	3463 (0.0996)	3462 (0.0997)
16	16	8	8	4	4	6154 (0.0499)	6‘99 (0.0499)	5515 (0.0998)	5545 (0.0997)
16	16	8	8	5	5	4121 (0.0498)	4235 (0.0498)	3611 (0.0999)	3675 (0.0999)

Table 8 : Empirical power of $C_1^*(b,d)$ for different values of m, n, m^+, n^+, b, d and various distributions for $\alpha = 0.10$

m	n	m^+	n^+	b	d	Distributions	σ						
							1.2	1.5	2	2.5	3	4	5
6	6	2	2	2	2	Uniform	0.111	0.208	0.276	0.324	0.345	0.398	0.416
						Normal	0.153	0.220	0.305	0.363	0.411	0.458	0.489
						Logistic	0.150	0.210	0.291	0.349	0.392	0.449	0.470
						Laplace	0.138	0.193	0.256	0.306	0.341	0.404	0.437
						Cauchy	0.129	0.159	0.204	0.229	0.267	0.311	0.344
6	6	3	3	2	2	Uniform	0.171	0.267	0.310	0.337	0.346	0.374	0.368
						Normal	0.216	0.255	0.310	0.342	0.359	0.375	0.375
						Logistic	0.208	0.243	0.293	0.327	0.339	0.352	0.361
						Laplace	0.196	0.230	0.273	0.317	0.315	0.340	0.339
						Cauchy	0.180	0.204	0.224	0.242	0.254	0.268	0.275
8	8	4	4	2	2	Uniform	0.120	0.214	0.281	0.317	0.333	0.373	0.390
						Normal	0.163	0.211	0.280	0.325	0.354	0.381	0.388
						Logistic	0.160	0.199	0.269	0.304	0.328	0.359	0.371
						Laplace	0.158	0.190	0.245	0.282	0.301	0.337	0.358
						Cauchy	0.128	0.162	0.183	0.207	0.218	0.235	0.247
16	16	8	8	2	2	Uniform	0.097	0.245	0.366	0.434	0.481	0.553	0.582
						Normal	0.162	0.261	0.393	0.475	0.534	0.592	0.612
						Logistic	0.158	0.236	0.350	0.439	0.487	0.551	0.582
						Laplace	0.142	0.208	0.292	0.362	0.428	0.501	0.533
						Cauchy	0.119	0.159	0.195	0.239	0.266	0.307	0.338
16	16	8	8	3	3	Uniform	0.099	0.234	0.302	0.338	0.349	0.380	0.370
						Normal	0.152	0.213	0.282	0.310	0.326	0.319	0.305
						Logistic	0.140	0.193	0.246	0.281	0.299	0.310	0.299
						Laplace	0.134	0.175	0.223	0.257	0.281	0.292	0.291
						Cauchy	0.115	0.127	0.144	0.151	0.164	0.181	0.183
16	16	8	8	5	5	Uniform	0.104	0.199	0.221	0.221	0.213	0.208	0.195
						Normal	0.134	0.159	0.180	0.185	0.175	0.154	0.142
						Logistic	0.121	0.155	0.169	0.167	0.167	0.153	0.143
						Laplace	0.118	0.139	0.160	0.169	0.168	0.162	0.148
						Cauchy	0.107	0.113	0.114	0.123	0.108	0.114	0.111

Table 9: Empirical power of $C_2^*(b,d)$ for different values of m, n, m^+, n^+, b, d and various distributions for $\alpha = 0.10$

m	n	m^+	n^+	b	d	Distributions	σ						
							1.2	1.5	2	2.5	3	4	5
6	6	2	2	2	2	Uniform	0.103	0.078	0.061	0.059	0.057	0.052	0.048
						Normal	0.9200	0.088	0.072	0.066	0.063	0.065	0.061
						Logistic	0.098	0.089	0.077	0.074	0.073	0.067	0.065
						Laplace	.101	0.095	0.085	0.086	0.075	0.074	0.070
						Cauchy	0.103	0.111	0.104	0.107	0.115	0.113	0.106
6	6	3	3	2	2	Uniform	0.174	0.179	0.172	0.173	0.176	0.179	0.191
						Normal	0.178	0.187	0.203	0.210	0.224	0.233	0.255
						Logistic	0.183	0.196	0.196	0.211	0.232	0.237	0.250
						Laplace	0.187	0.193	0.200	0.219	0.224	0.237	0.247
						Cauchy	0.1920	0.206	0.228	0.237	0.251	0.266	0.286
8	8	4	4	2	2	Uniform	0.097	0.098	0.094	0.102	0.107	0.118	0.120
						Normal	0.105	0.118	0.130	0.138	0.155	0.172	0.188
						Logistic	0.108	0.120	0.138	0.155	0.154	0.181	0.193
						Laplace	0.107	0.120	0.137	0.146	0.162	0.175	0.183
						Cauchy	0.111	0.131	0.152	0.171	0.184	0.205	0.217
16	16	8	8	2	2	Uniform	0.106	0.107	0.115	0.127	0.125	0.137	0.158
						Normal	0.124	0.140	0.168	0.198	0.223	0.270	0.300
						Logistic	0.119	0.149	0.177	0.205	0.239	0.271	0.308
						Laplace	0.123	0.149	0.183	0.197	0.219	0.261	0.292
						Cauchy	0.135	0.162	0.209	0.246	0.277	0.317	0.366
16	16	8	8	3	3	Uniform	0.099	0.097	0.087	0.082	0.086	0.083	0.083
						Normal	0.108	0.108	0.104	0.106	0.108	0.116	0.112
						Logistic	0.110	0.113	0.120	0.113	0.121	0.118	0.119
						Laplace	0.104	0.119	0.126	0.124	0.130	0.140	0.133
						Cauchy	0.111	0.124	0.145	0.154	0.173	0.181	0.183
16	16	8	8	5	5	Uniform	0.099	0.093	0.087	0.079	0.081	0.072	0.069
						Normal	0.092	0.093	0.088	0.081	0.078	0.073	0.064
						Logistic	0.096	0.096	0.091	0.090	0.082	0.076	0.073
						Laplace	0.093	0.104	0.100	0.093	0.094	0.095	0.083
						Cauchy	0.097	0.106	0.115	0.110	0.113	0.109	0.105

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